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Stability Analysis of Numerial Methods for Stochastic Differential Equations

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Abstract. Stochastic differential equations (SDEs) represent physical phenomena dominated by stochastic processes. Like as for deterministic differential equations (DDEs), various numerical methods are proposed for SDEs. In this note we study stability of numerical methods for scalar stochastic differential equation with regard to the mean-square norm. This notion is an extention of absolute stability in numerial methods for DDEs.

1. Introduction

We consider stochastic initial value problem (SIVP) for scalar autonomous Ito stochastic differential equation given by

$$\begin{cases} dX(t) = f(X)dt + g(X)dW(t), & t \in [0,T], \\ X(0) = x, \end{cases} \tag{1}$$

where $W(t)$ represents the standard Wiener process and initial value x is a fixed value. Some authors (e.g. in [1, 3, 5, 6, 7, 8, 9, 10, 11, 12]) propose various numerical schemes for SDE (1), which recursively compute sample paths (trajectories) of solution $X(t)$ at step-points. Numerical experiments for these shemes have appeared in some papers([5, 7, 8, 12]). However numerical stability of the schemes is analyzed in quite few papers. We will propose stochastic version of absolute stability analysis under the mean-square norm. We will show stability regions of various numerical schemes in case of test equation under our notion. This is an extention of absolute stability in numerical methods for ordinary differential equations (ODEs).

2. Numerical schemes for SDEs

Here we present some numerical schemes for SDEs. They adopt an equidistant discretization of the time interval $[0,T]$ with stepsize

$$h = \frac{T}{N} \quad \text{for fixed natural number } N.$$

Furthermore,

$$t_n = nh, \qquad n \in \{1, 2, \cdots, N\}$$

denotes the n-th step-point. We abbreviate

$$\bar{X}_n = \bar{X}(t_n) \quad \text{and} \quad \Phi_n = \Phi(\bar{X}_n),$$

for all $n \in \{0, \cdots, N\}$ and functions $\Phi : \mathbf{R} \mapsto \mathbf{R}$. When $X(t)$ and $\bar{X}_n$ stand for the exact and the numerical solutions of SIVP (1), respectively, the local error from $t = t_{n-1}$ to $t = t_n$ and the global error from $t = t_0$ to $t = T = t_N$ are defined by the following:

$$\mathbf{E}(|X(t_n) - \bar{X}_n|^2 | X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1}),$$

$$\mathbf{E}(|X(T) - \bar{X}_N|^2 | X_0 = \bar{X}_0 = \bar{x}_0),$$

where $\bar{x}_{n-1}$, $\bar{x}_0$ are arbitrary real values. Then, the local and global orders are defined as follows.

Definition 1 *The numerical scheme $\bar{X}_n$ is of local order γ, of global order β iff*

$$\mathbf{E}(|X(t_n) - \bar{X}_n|^2 | X(t_{n-1}) = \bar{X}_{n-1} = \bar{x}_{n-1}) = O(h^{\gamma+1}) \qquad (h \downarrow 0),$$

$$\mathbf{E}(|X(T) - \bar{X}_N|^2 | X_0 = \bar{X}_0 = \bar{x}_0) = O(h^{\beta}) \qquad (h \downarrow 0),$$

respectively.

Remark. While the equation $\gamma = \beta$ holds in numerical methods for ODE under a mild assumption, it isn't satisfied for SDE (See [11]). Also another definition of order of convergence may be given by

$$\mathbf{E}(|X(T) - \bar{X}_N| | X_0 = \bar{X}_0 = \bar{x}_0) = O(h^{\beta'}) \qquad (h \downarrow 0)$$

to be consistent with the deterministic order of convergence ([3]). But we use the order concept in Definition 1 to make it easy to investigate the global error. Thus the reader might read as $\beta = 2\beta'$. □

The following three random variables will be used in the $(n+1)$-st time step of the schemes:

$$\Delta W_n = W(t_{n+1}) - W(t_n),$$

$$\Delta Z_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dW(r) ds,$$

$$\Delta \bar{Z}_n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} dr dW(s).$$

They are obtained as sample values of normal random variables using the transformation

$$\Delta W_n = \xi_{n,1} h^{1/2},$$

$$\Delta Z_n = \frac{1}{2}(\xi_{n,1} + \frac{\xi_{n,2}}{\sqrt{3}})h^{3/2},$$
$$\Delta \bar{Z}_n = \frac{1}{2}(\xi_{n,1} - \frac{\xi_{n,2}}{\sqrt{3}})h^{3/2}$$

and, together with them, we further use

$$\Delta \tilde{W}_n = \xi_{n,2}h^{1/2},$$

where $\xi_{n,1}$, $\xi_{n,2}$ are mutually independent $N(0,1)$ random variables.

Remark. In mean-square sense ΔZ_n and $\Delta \bar{Z}_n$ cannot be expressed in terms of the independent $N(0,1)$ random variables $\xi_{n,1}$, $\xi_{n,2}$, $\cdots$, $\xi_{n,m}$. Thus any numerical scheme cannot attain order 3 ([7],[8],[10]). However we derived the above expressions for ΔZ_n and $\Delta \bar{Z}_n$ in the weak sense. In the simulation on digital computer with pseudo-random numbers we might expect these random variables behaves well for the approximate solution. □

Numerical schemes

1. $\gamma = 1$, $\beta = 1$.

i) Euler-Maruyama scheme (Maruyama 1955, see [1]):

$$\bar{X}_{n+1} = \bar{X}_n + f_n h + g_n \Delta W_n. \tag{2}$$

2. $\gamma = 2$, $\beta = 2$.

i) Heun scheme (McShane 1974, see [1]):

$$\bar{X}_{n+1} = \bar{X}_n + \frac{1}{2}[F_1 + F_2]h + \frac{1}{2}[G_1 + G_2]\Delta W_n, \tag{3}$$

where

$$\begin{aligned} F_1 &= F(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ F_2 &= F(\bar{X}_n + F_1 h + G_1 \Delta W_n), \\ G_2 &= g(\bar{X}_n + F_1 h + G_1 \Delta W_n), \end{aligned}$$
$$F(x) = [f - \frac{1}{2}g'g](x).$$

ii) Taylor scheme (Mil'shtein 1974 [6]):

$$\bar{X}_{n+1} = \bar{X}_n + f_n h + g_n \Delta W_n + \frac{1}{2}[g'g]_n((\Delta W_n)^2 - h). \tag{4}$$

iii) Derivative-free scheme (Platen 1984, see [3, 5]):

$$\bar{X}_{n+1} = \bar{X}_n + F_1 h + G_1 \Delta W_n + [G_2 - G_1] h^{-\frac{1}{2}} \frac{(\Delta W_n)^2 - h}{2}, \qquad (5)$$

where

$$\begin{aligned} F_1 &= f(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ G_2 &= g(\bar{X}_n + G_1 h^{1/2}). \end{aligned}$$

iv) FRKI method (Newton 1991 [7]))

$$\bar{X}_{n+1} = \bar{X}_n + F_1 h + G_2 \Delta W_n + [G_2 - G_1] h^{\frac{1}{2}}, \qquad (6)$$

where

$$\begin{aligned} F_1 &= f(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ G_2 &= g(\bar{X}_n + G_1(\Delta W_n - h^{1/2})/2). \end{aligned}$$

v) Implicit schem (Platen) :

$$\bar{X}_{n+1} = \bar{X}_n + \{\alpha f_{n+1} + (1-\alpha) f_n\} h + g_n \Delta W_n + \frac{1}{2}[g'g]_n((\Delta W_n)^2 - h). \quad (7)$$

Here the scheme is particularly called trapezoidal scheme if $\alpha = 1/2$; backward Euler scheme if $\alpha = 1$. Clearly (7) is identical to Taylor scheme (4) if $\alpha = 0$.

3. $\gamma = 3$, $\beta = 2$.
i) Improved 3-stage RK scheme ([11]):

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \tfrac{1}{4}[F_1 + 3F_3] h + \tfrac{1}{4}[G_1 + 3G_3] \Delta W_n \\ &\quad + \tfrac{1}{2\sqrt{3}}[f'g - g'f - \tfrac{1}{2} g'' g^2]_n h \Delta \tilde{W}_n, \end{aligned} \qquad (8)$$

where

$$\begin{aligned} F_1 &= F(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ F_2 &= F(\bar{X}_n + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1 \Delta W_n), \\ G_2 &= g(\bar{X}_n + \tfrac{1}{3}F_1 h + \tfrac{1}{3}G_1 \Delta W_n), \\ F_3 &= F(\bar{X}_n + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2 \Delta W_n), \\ G_3 &= g(\bar{X}_n + \tfrac{2}{3}F_2 h + \tfrac{2}{3}G_2 \Delta W_n), \end{aligned}$$

$$F(x) = [f - \frac{1}{2}g'g](x).$$

ii) Taylor scheme (Mil'shtein 1974 [6]):

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + f_n h + g_n \Delta W_n \\ &+\tfrac{1}{2}[g'g]_n((\Delta W_n)^2 - h) \\ &+[f'g]_n \Delta Z_n \\ &+[g'f + \tfrac{1}{2}g''g^2]_n \Delta \bar{Z}_n \\ &+\tfrac{1}{6}[g'^2 g + g''g^2]_n((\Delta W_n)^3 - 3h\Delta W_n). \end{aligned} \tag{9}$$

4. $\gamma = 3$, $\beta = 3$.
i) Taylor scheme (Platen [3]):

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + f_n h + g_n \Delta W_n \\ &+\tfrac{1}{2}[g'g]_n((\Delta W_n)^2 - h) \\ &+[f'g]_n \Delta Z_n \\ &+[g'f + \tfrac{1}{2}g''g^2]_n \Delta \bar{Z}_n \\ &+\tfrac{1}{6}[g'^2 g + g''g^2]_n((\Delta W_n)^3 - 3h\Delta W_n) \\ &+\tfrac{1}{2}[f'f + \tfrac{1}{2}f''g^2]_n h^2. \end{aligned} \tag{10}$$

5. $\gamma = 2$, $\beta = 2$ but $\gamma = 3$, $\beta = 3$ for linear equation.
i) ERKI method (Newton 1991 [7])

$$\begin{aligned} \bar{X}_{n+1} &= \bar{X}_n + \tfrac{1}{2}[F_1 + F_2]h + \tfrac{1}{40}[37G_1 + 30G_3 - 27G_4]\Delta W_n \\ &\quad + \tfrac{1}{16}[8G_1 + G_2 - 9G_3]\sqrt{3h}, \end{aligned} \tag{11}$$

where

$$\begin{aligned} F_1 &= f(\bar{X}_n), \\ G_1 &= g(\bar{X}_n), \\ F_2 &= f(\bar{X}_n + F_1 h + G_1 \Delta W_n), \\ G_2 &= g(\bar{X}_n - \tfrac{2}{3}G_1(\Delta W_n + \sqrt{3h})), \\ G_3 &= g(\bar{X}_n + \tfrac{2}{9}G_1(3\Delta W_n + \sqrt{3h})), \\ G_4 &= g(\bar{X}_n - \tfrac{20}{27}F_1 h + \tfrac{10}{27}(G_2 - G_1)\Delta W_n - \tfrac{10}{27}G_2\sqrt{3h}), \end{aligned}$$

3. Linear stability analysis

We consider the Ito's test equation (supermartingale eqn.) with real numbers $\lambda < 0$ and $\mu \geq 0$,

$$\begin{cases} dX(t) = \lambda X dt + \mu X dW(t), & \\ X(0) = 1, & t \in [0, T] \end{cases} \tag{12}$$

where the exact solution of (12) is

$$X(t) = \exp\{(\lambda - \frac{1}{2}\mu^2)t + \mu W(t)\}. \tag{13}$$

From qualitative theory of SDEs $X(t) \equiv 0$ is stochastically asymptotically stable in the large if $\lambda - \frac{1}{2}\mu^2 < 0$ and unstable if $\lambda - \frac{1}{2}\mu^2 \geq 0$ for (12) ([1]). In a few papers (e.g. [8]) we can see numerical stability in a similar sense as above. However, the analytical theory cannot be applicable to numerical schemes, because it is impossible to carry out a numerical scheme until all the sample paths (13) diminish to 0 if $\lambda > 0$ and $\lambda - \frac{1}{2}\mu^2 < 0$. This implies that some sample paths decrese to 0, whereas their distributions increse. Thus we study only SDE having all sample paths whose distribution tends to 0 as $t \to \infty$. Now let us study the condition

$$\|X(t)\| \to 0 \qquad \text{as } t \to \infty \tag{14}$$

where $\|\cdot\|$ denotes
$$\|X\| = \{\mathbf{E}|X|^2\}^{\frac{1}{2}}.$$

Lemma 1 *For the solution of test equation (12) , (14) iff*
$$2\lambda + \mu^2 < 0. \tag{15}$$

Proof. Assume $Y(t) = \mathbf{E}|X(t)|^2$, then $Y(t)$ satisfies the following ordinary differential equation
$$\begin{cases} dY = (2\lambda + \mu^2)Y dt, & \\ Y(0) = 1, & t \in [0, T] \end{cases} \tag{16}$$

which has the solution $Y(t) = \exp\{(2\lambda + \mu^2)t\}$. Therefore the condition (14) turns out to (15). Conversely, it follows from (15) that solution $X(t)$ in (13) satisfy (14). □

Of couse, under the assumption (15) $X(t) \equiv 0$ in Lemma 1 is stochastically asymptotically stable in the large, because $2\lambda - \mu^2 \leq 2\lambda + \mu^2 < 0$.

We now ask what conditions must be imposed in order that the numerical solution $\{\bar{X}_n\}$ of (12) generated by a numerical scheme satisfy
$$\|\bar{X}_n\| \to 0 \qquad \text{as } n \to \infty. \tag{17}$$

We now define $\bar{Y}_n$ corresponding to $Y(t)$ by
$$\bar{Y}_n = \mathbf{E}|\bar{X}_n|^2. \tag{18}$$

When we apply numerical scheme to (12) and take the mean-square norm, we obtain a one-step difference equation of the form
$$\bar{Y}_{n+1} = R(\bar{h}, k)\bar{Y}_n \tag{19}$$

where $\bar{h} = h\lambda$ and $k = -\mu^2/\lambda$. We shall call $R(\bar{h}, k)$ the *stability function* of the scheme. Clearly $\bar{Y}_n \to 0$ as $n \to \infty$ iff
$$|R(\bar{h}, k)| < 1. \tag{20}$$

Hence the scheme is said to be absolutely stable for those values of $\bar{h}$ satisfying (20). The region $\mathcal{R}$ given by
$$\mathcal{R} = \{(\bar{h}, k);\ (20)\,\text{holds}\}$$

is analogously called the region of absolute stability of the scheme.

We will calculate the stability function of various schemes listed in the last section, whose region of absolute stability is given in the end of this note.

4. Stability regions of schemes

1. i) Euler-Maruyama scheme (2)

$$R(\bar{h},k) = (1+\bar{h})^2 - k\bar{h} \tag{21}$$

2. i) Heun scheme (3)

First of all, to make it easy to derive the stability function R of Heun scheme, we consider stability function R' corresponding to Stratonovich equation

$$\begin{cases} dX(t) = \lambda' X dt + \mu X \circ dW(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad t \in [0,T]. \\ X(0) = 1, \quad \lambda' = \lambda - \frac{1}{2}\mu^2, \end{cases} \tag{22}$$

Due to the shift of Ito SDE to Stratonovich one, the function $F(x)$ in the scheme (3) reduces to $f(x)$ itself. Let $\bar{h}' = h\lambda'$ and $k' = -\mu^2/\lambda'$, Stratonovich stability function R' is

$$\begin{aligned} R'(\bar{h}',k') &= \{1+(1-\tfrac{1}{2}k')\bar{h}' + \tfrac{1}{2}\bar{h}'^2\}^2 - (1+\bar{h}')^2 k'\bar{h}' \\ &\quad + \tfrac{1}{2}(k'\bar{h}')^2. \end{aligned} \tag{23}$$

Thus, the transformation $\bar{h}' = (1+k/2)\bar{h}$ and $k' = 2k/(2+k)$ leads to Ito stability function R as

$$R(\bar{h},k) = R'((1+\frac{1}{2}k)\bar{h}, \frac{2k}{2+k}). \tag{24}$$

ii) Taylor scheme (4), iii) Derivative-free scheme (5) and iv) FRKI method (6)

$$R(\bar{h},k) = (1+\bar{h})^2 - k\bar{h} + \frac{1}{2}k^2\bar{h}^2 \tag{25}$$

v) Implicit scheme (7)

$$R(\bar{h},k,\alpha) = \frac{\{1+(1-\alpha)\bar{h}\}^2 - k\bar{h} + \frac{1}{2}k^2\bar{h}^2}{(1-\alpha\bar{h})^2} \tag{26}$$

Particularly, (a) trapezoidal scheme ($\alpha = 1/2$)

$$R(\bar{h},k) = \frac{(1+\frac{1}{2}\bar{h})^2 - k\bar{h} + \frac{1}{2}k^2\bar{h}^2}{(1-\frac{1}{2}\bar{h})^2} \tag{27}$$

(b) backward Euler scheme ($\alpha = 1$)

$$R(\bar{h}, k) = \frac{1 - k\bar{h} + \frac{1}{2}k^2\bar{h}^2}{(1 - \bar{h})^2} \tag{28}$$

3. i) Improved 3-stage RK scheme (8)

$$\begin{aligned} R'(\bar{h}', k') &= (1 + \bar{h}' + \tfrac{1}{2}\bar{h}'^2 + \tfrac{1}{6}\bar{h}'^3)^2 \\ &\quad -(1 + \bar{h}' + \tfrac{1}{2}\bar{h}'^2)^2 k'\bar{h}' \\ &\quad +(1 + \bar{h}')\tfrac{3}{4}k'^2\bar{h}' - \tfrac{15}{36}k'^3\bar{h}'^3 \\ &\quad -(1 + \bar{h}' + \tfrac{1}{2}\bar{h}'^2 + \tfrac{1}{6}\bar{h}'^3)(1 + \bar{h}')k'\bar{h}' \\ &\quad +(1 + \bar{h}' + \tfrac{1}{2}\bar{h}'^2)k'^2\bar{h}'^2 \end{aligned} \tag{29}$$

$$R(\bar{h}, k) = R'((1 + \frac{1}{2}k)\bar{h}, \frac{2k}{2 + k}) \tag{30}$$

Again the shift to Stratonovich SDE has been utilized.
ii) Taylor scheme (9)

$$R(\bar{h}, k) = (1 + \bar{h})^2 - (1 + \bar{h})^2 k\bar{h} + \frac{1}{2}k^2\bar{h}^2 - \frac{1}{6}k^3\bar{h}^3 \tag{31}$$

4. i) Taylor scheme (10) and 5. i) ERKI method (11)

$$R(\bar{h}, k) = (1 + \bar{h} + \frac{1}{2}\bar{h}^2)^2 - (1 + \bar{h})^2 k\bar{h} + \frac{1}{2}k^2\bar{h}^2 - \frac{1}{6}k^3\bar{h}^3 \tag{32}$$

From Figs 1-8 we can conclude that backward Euler scheme is superior in stability to other schemes. In particular for implicit scheme the following lemma holds.

Lemma 2 *Implicit scheme (7) of* $1/2 \le \alpha \le 1$ *is unconditionally stable for* $0 \le k \le \sqrt{4\alpha - 2}$.

Proof. The stability function of implicit scheme was given by the following expression:

$$R(\bar{h}, k, \alpha) = \frac{\{1 + (1 - \alpha)\bar{h}\}^2 - k\bar{h} + \frac{1}{2}k^2\bar{h}^2}{(1 - \alpha\bar{h})^2}.$$

If $0 \le \alpha \le 1/2$, the boundary of the region, that is $|R(\bar{h}, k, \alpha)| = 1$ intesects with $\bar{h}$-axis at the point $(0, 2/(2\alpha - 1))$. On the other hand, if

$1/2 \leq \alpha \leq 1$, the boundary of the region doesn't intersect with $\bar{h}$-axis and has an asymptote $k = \sqrt{4\alpha - 2}$. Therefore the result follows. □

Also the region of 2nd order Taylor scheme (4) is smaller than Euler-Maruyama scheme (2). The RK scheme (3) and (8) are inferior to the other schemes of the same order. This is caused by the transformation $f - g'g/2$.

5. Numerical results

We show a result which confirms the analysis described in the previous section. We carry out Euler-Maruyama and backward Euler schemes for the test equation (12). As an example we select $(k, \bar{h}) = (1, -0.5)$ and $(1, -1)$, that is (i) $(\lambda, \mu, h) = (-100, 10, 0.005)$ and (ii) $(-100, 10, 0.01)$. Here Euler-Maruyama scheme is stable for the triplet (i), while unstable for (ii). On the other hand backward Euler scheme is stable in both (i) and (ii). We take 20,000 samples of pseudo-random number.

Table 1

	$\|X\|^2$			
	Euler-Maruyama		backward Euler	
t	(i)	(ii)	(i)	(ii)
0.010	0.5660	1.0140	0.5300	0.6495
0.020	0.3289	0.9955	0.3052	0.3982
0.030	0.1680	1.0696	0.1166	0.3093
0.040	0.1020	0.9826	0.0673	0.1719
0.050	0.0444	0.9631	0.0235	0.0547
0.060	0.0305	0.9724	0.0088	0.0225
0.070	0.0137	0.7908	0.0044	0.0145
0.080	0.0132	0.5820	0.0008	0.0104
0.090	0.0008	0.4826	0.0004	0.0014
0.100	0.0001	0.7225	0.0000	0.0013

(Macintosh SE/30)

8 Conclusions and future aspects

We can say that linear stability analysis for scalar SDE with real coefficients established, but we impose a strong condition, that is, we adopt the mean-square norm $\|X\| = \{\mathrm{E}|X|^2\}^{\frac{1}{2}}$. We are further required to consider linear satability for scalar SDE with complex coefficients and to extend it to multi-dim SDE or for multi-dim Wiener processes. Also we are interested in stability analysis based on Lyapunov exponent of SDE ([12]).

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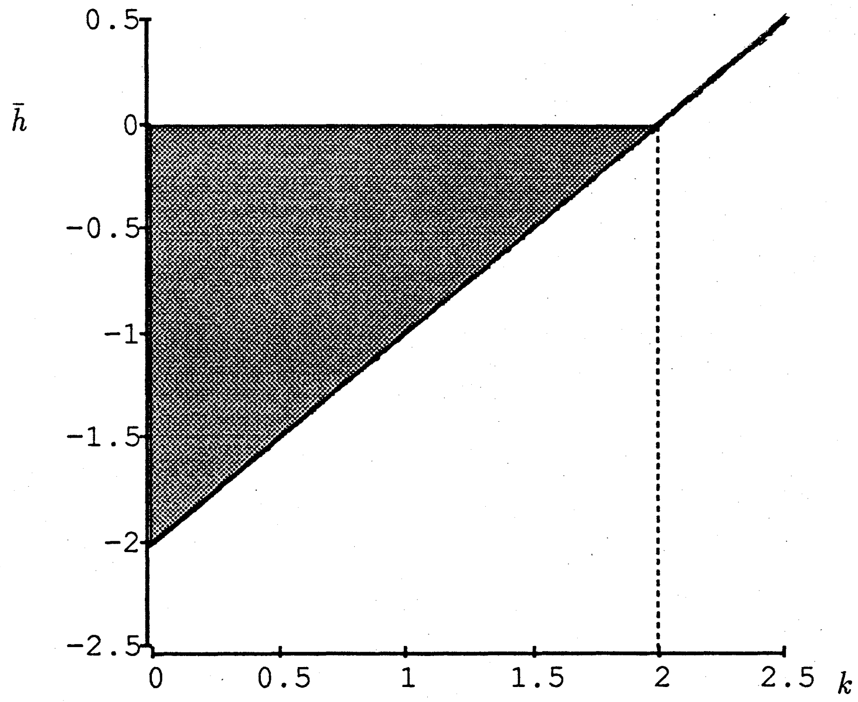

Fig 1. The region of absolute stability of scheme (2)

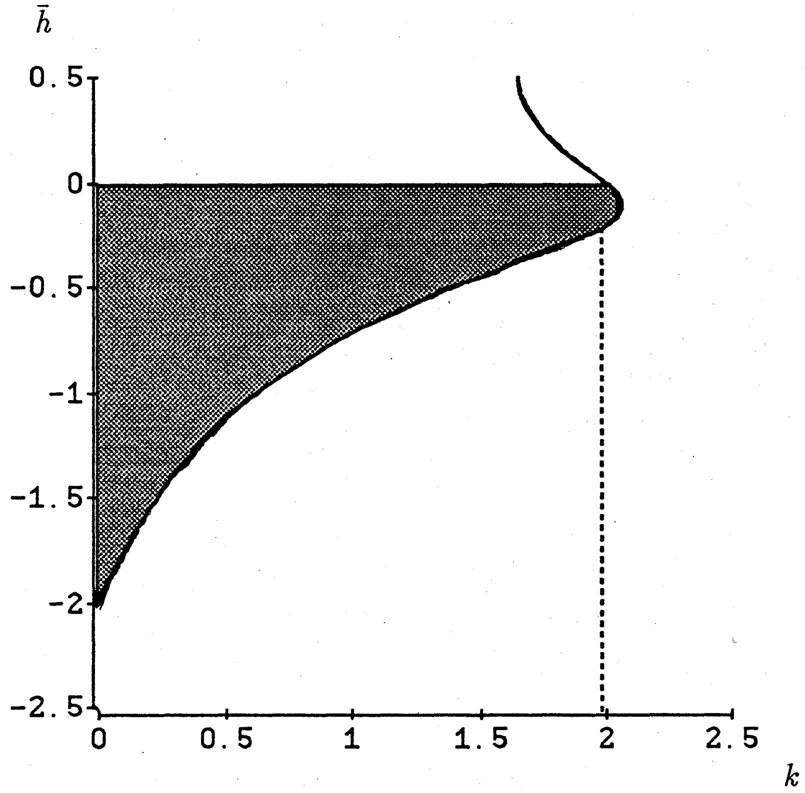

Fig 2. The region of absolute stability of scheme (3)

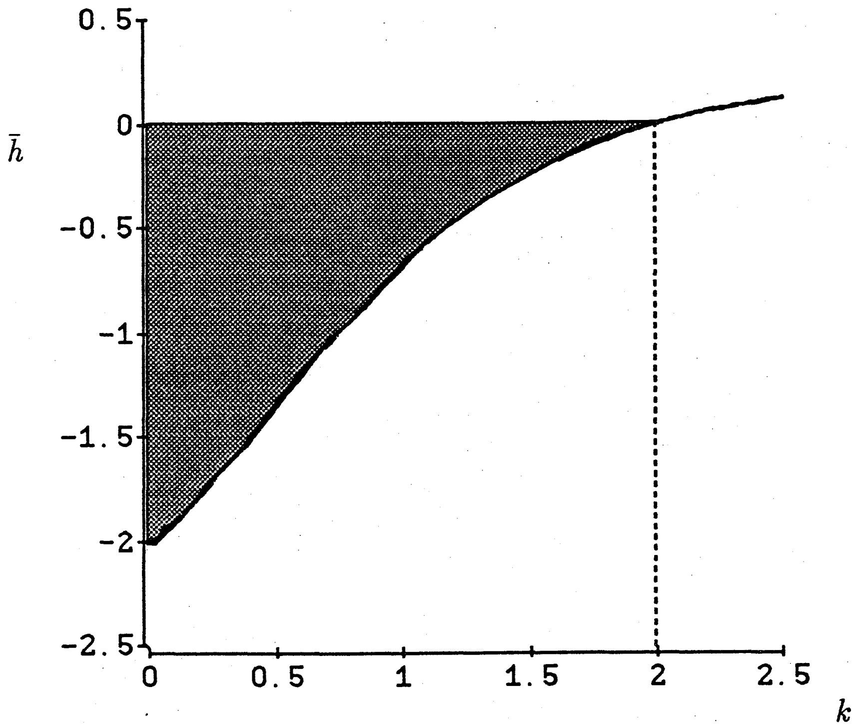

Fig 3. The region of absolute stability of schemes (4), (5) and (6)

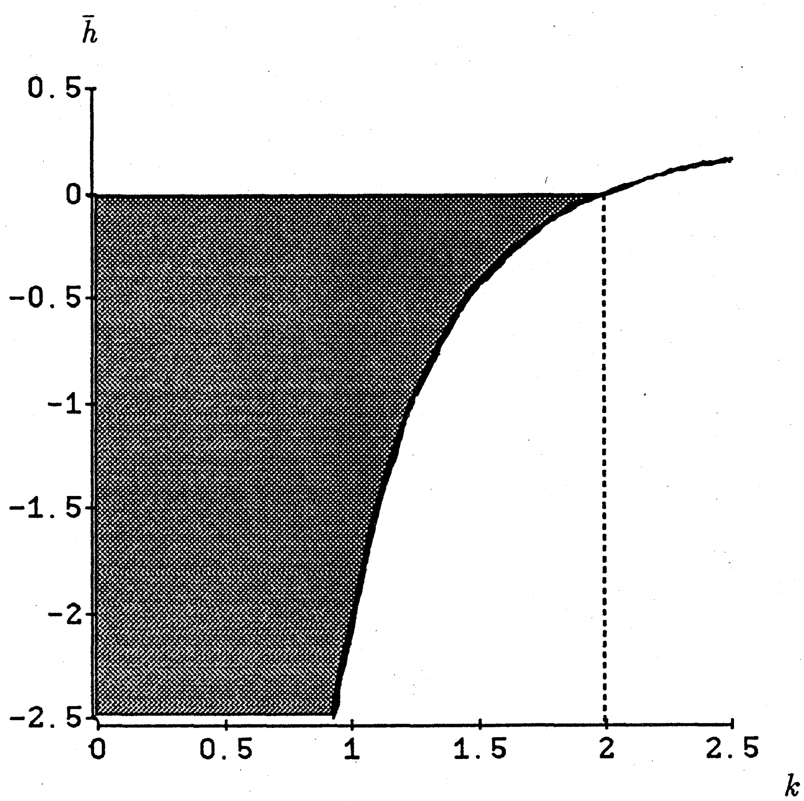

Fig 4. The region of absolute stability of trapezoidal scheme

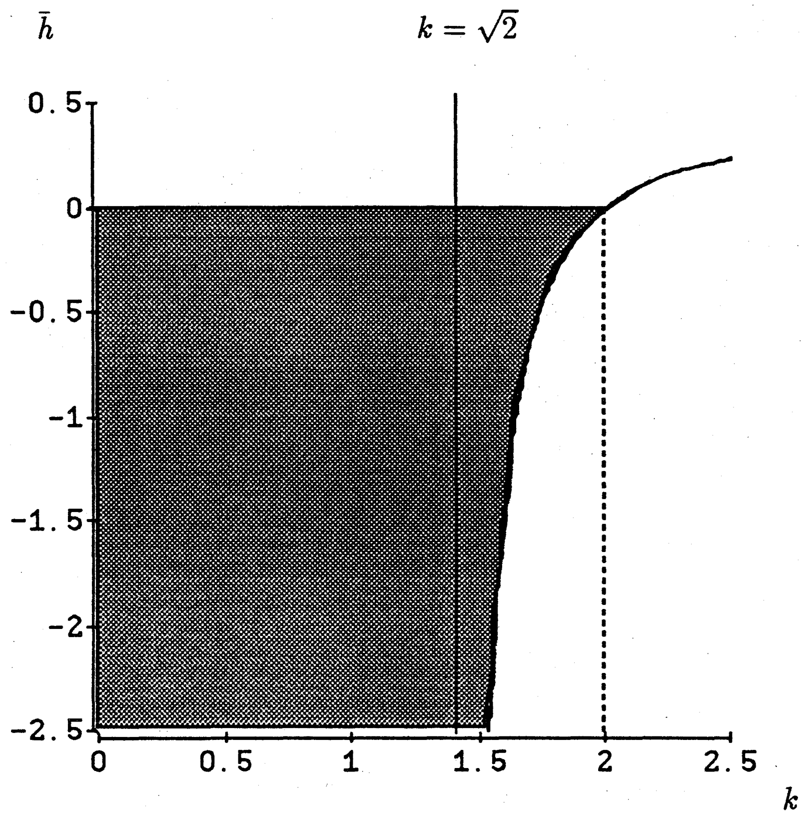

Fig 5. The region of absolute stability of backward Euler scheme

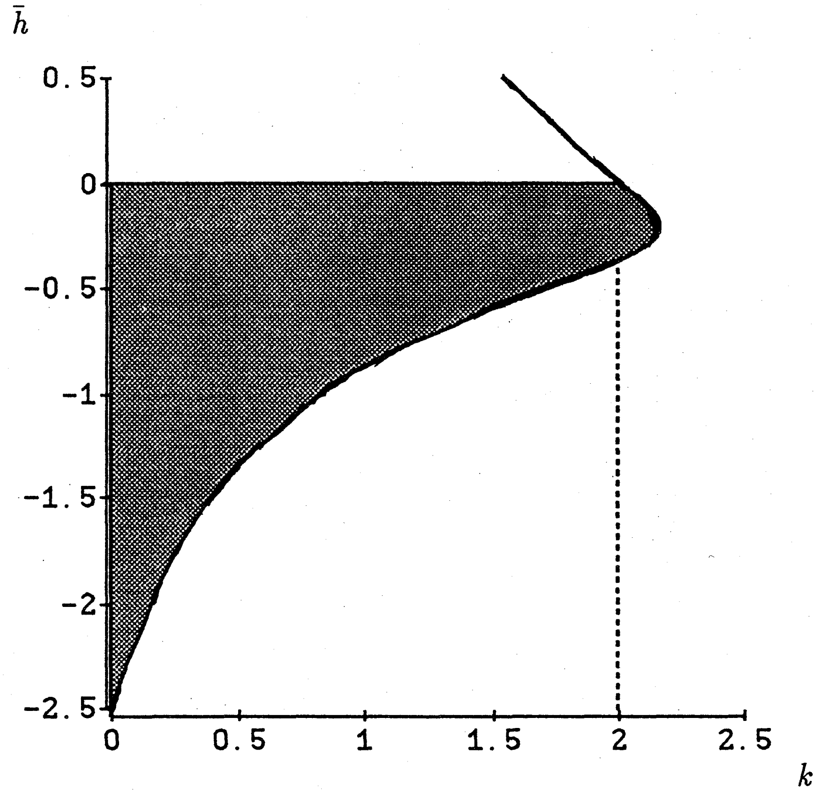

Fig 6. The region of absolute stability of scheme (8)

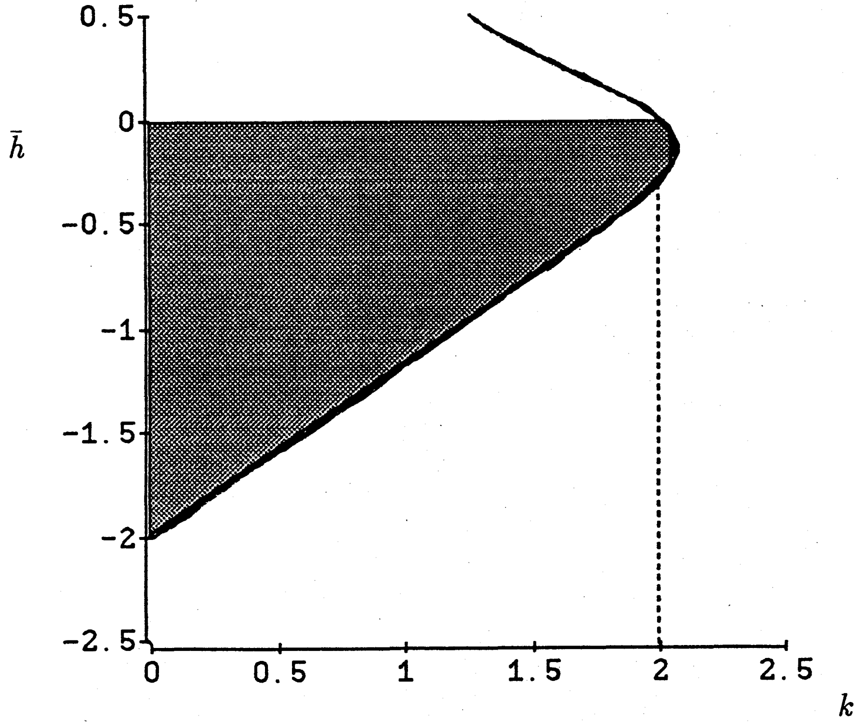

Fig 7. The region of absolute stability of scheme (9)

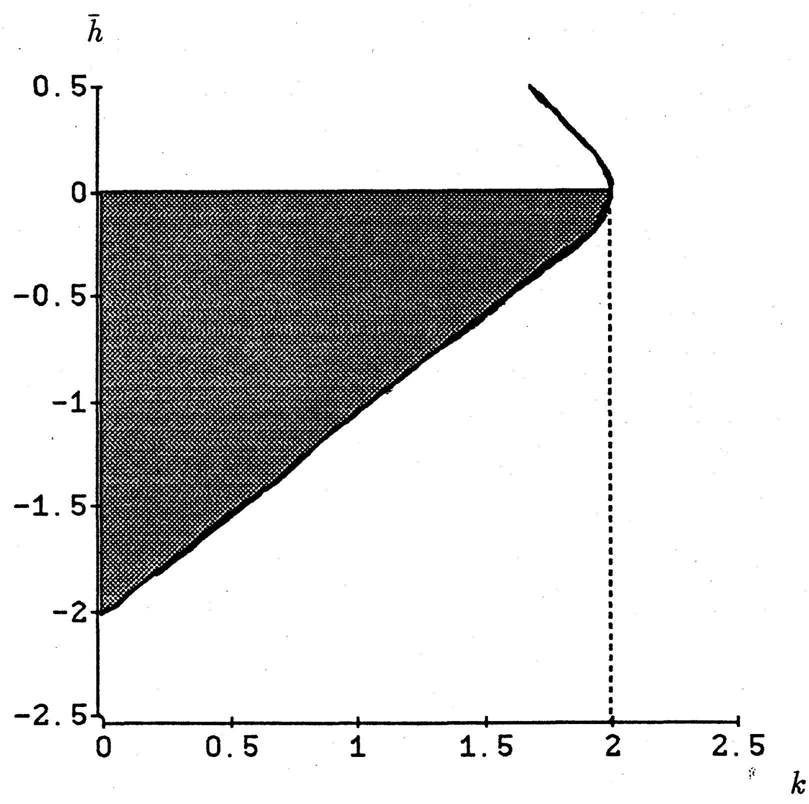

Fig 8. The region of absolute stability of scheme (10) and (11)